The effective potential and resummation procedure to multidimensional complex cubic potentials for weak and strong-coupling

S.-A. Yahiaoui, O. Cherroud, M. Bentaiba[1]

LPTHIRM, Département de Physique, Faculté des Sciences,

Université Saad DAHLAB de blida, Algérie.

Abstract

The method for the recursive calculation of the effective potential is applied successfully in case of weak coupling limit (g tend to zero) to a multidimensional complex cubic potential. In strong-coupling limit (g tend to infinity), the result is resumed using the variational perturbation theory (VPT). It is found that the convergence of VPT-results approaches those expected.

1 Introduction

Quantum dynamics allows to avoid the operator formalism for the benefit of infinite products of integrals, called Path-Integrals [1,2]. Unfortunately, most Path-Integrals can not be performed exactly; therefore, many different approximation procedures are developed in order to deal with non-analytically solvable systems.

The most commonly used one is known as perturbation theory [3-9]. It is based upon the expansion of some physical quantity, e.g. the ground-state energy of particle in some potential, into a power series of coupling constant, the results obtained in weak-coupling limit seem to converge

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[1]Corresponding author

E-mail address : bentaiba@hotmail.com

to the exact result for low orders, so that the divergence of that series becomes important when the expansion is driven to the higher-order [3-12]. Therefore, it is necessary to find means for treating the divergent perturbation series, including the strong-coupling limit. Thus, we apply the resummation scheme often based on introducing an artificial parameter trick, namely Kleinert's square-root [2,13-15].

The variational perturbation theory (VPT) [2,7-9] has been developed thanks to the variational approach due to R. P. Feynman and H. Kleinert [6], and has been extended to an efficient non-perturbation approximation. VPT allows the conversion of the divergent weak-coupling into the convergent strong-coupling expansion.

According to some studies, a more physical alternative axiom called space-time reflection symmetry ($\mathcal{PT}-$symmetry) allows for possibility of cubic complex non-Hermitian Hamiltonian but still leads to a consistant theory of quantum mechanics. It is shown that if the $\mathcal{PT}-$symmetry of a Hamiltonian H is not broken, then the accompanying spectrum remains real and positive.

The space reflection operator $\mathcal{P}$ is a linear operator with the property $\mathcal{P}^2 = 1$ and has the effects $p \to -p$, and $x \to -x$, while the time reflection $\mathcal{T}$ is an antilinear operator with the property $\mathcal{T}^2 = 1$ and has the effects $p \to -p$, $x \to x$, and $i \to -i$.

Recently, the convergence of VPT has been tested successfully for the ground-state energy of the one-dimensional complex cubic coupled oscillator [11,13,15]

$$V(x) = \frac{M}{2}\omega^2 x^2 + igx^3. \tag{1}$$

The purpose of this paper is to examine how, in a first approach, the VPT can be applied to the generalized 2D and 3D-complex cubic potentials

$$V(x,y) = \frac{M}{2}\omega^2 \left(x^2 + y^2\right) + igxy^2. \tag{2}$$

$$V(x,y,z) = \frac{M}{2}\omega^2 \left(x^2 + y^2 + z^2\right) + igxyz. \tag{3}$$

by resuming the weak-coupling series [10,12] of the ground-state energy via VPT.

In the strong-coupling limit, the potentials (2) and (3) are reduced to potentials without a harmonic term. It turns out that the rate of convergence is not satisfactory. Therefore, combining the effective potential [13-15] to the VPT permits the improvement of the rate of convergence. This later should be performed to higher orders, and since mathematical computation become cumbersome, we have stopped our calculation at the second order in $\hbar$.

2 Perturbation theory : Feynman diagrams

In this section, we derive the weak-coupling coefficients for the ground-state energy of the potentials (2) and (3) using the Feynman diagrammatical expansion. The partition functions

$$Z_{2D} = \oint \mathcal{D}x\mathcal{D}y \exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left[\frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{M\omega^2}{2}\left(x^2 + y^2\right) + igxy^2\right]\right\},$$

(4)

$$Z_{3D} = \oint \mathcal{D}x\mathcal{D}y\mathcal{D}z \exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left[\frac{M}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right.\right.$$
$$\left.\left. +\frac{M\omega^2}{2}\left(x^2 + y^2 + z^2\right) + igxyz\right]\right\}.$$

(5)

will be calculated perturbatively by an expansion in the coupling constant g, here we have used x_α (resp. $\dot{x}_\alpha$) instead of $x_\alpha(\tau)$ (resp. $\dot{x}_\alpha(\tau)$), and dot " $\cdot$ " refers to derivatives of $x_\alpha(\tau)$ with respect to τ. These expressions can be converted, using the Taylor expansion of the exponential function, into the form

$$Z_{2D} = \oint \mathcal{D}x\mathcal{D}y \exp\left\{-\frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau \left[\frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{M\omega^2}{2}\left(x^2 + y^2\right)\right]\right\}$$

$$\times \left(1 - \frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau_1 \left[igx\left(\tau_1\right)y^2\left(\tau_1\right)\right]\right.$$

$$\left. +\frac{1}{2\hbar^2}\int\limits_0^{\hbar\beta} d\tau_1 \int\limits_0^{\hbar\beta} d\tau_2 \left[igx\left(\tau_1\right)y^2\left(\tau_1\right)\right]\left[igx\left(\tau_2\right)y^2\left(\tau_2\right)\right] + \cdots\right), \quad (6)$$

$$Z_{3D} = \oint \mathcal{D}x\mathcal{D}y\mathcal{D}z \exp\left\{-\frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau \left[\frac{M}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{M\omega^2}{2}\left(x^2 + y^2 + z^2\right)\right]\right\}$$

$$\times \left(1 - \frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau_1 \left[igx\left(\tau_1\right)y\left(\tau_1\right)z\left(\tau_1\right)\right] + \right.$$

$$\left. \frac{1}{2\hbar^2}\int\limits_0^{\hbar\beta} d\tau_1 \int\limits_0^{\hbar\beta} d\tau_2 \left[igx\left(\tau_1\right)y\left(\tau_1\right)z\left(\tau_1\right)\right]\left[igx\left(\tau_2\right)y\left(\tau_2\right)z\left(\tau_2\right)\right] + \cdots\right). \quad (7)$$

Introducing in both cases the notation, namely expectation values,

$$\langle\cdots\rangle_\omega \equiv \frac{1}{\prod\limits_{\alpha=1}^3 Z_{x_\alpha}^\omega} \oint \prod_{\alpha=1}^3 \mathcal{D}x_\alpha \left(\cdots\right) \exp\left\{-\frac{1}{\hbar}\int\limits_0^{\hbar\beta} d\tau \sum_{\alpha=1}^3 \left(\frac{M}{2}\dot{x}_\alpha^2 + \frac{M}{2}x_\alpha^2\right)\right\},$$

$$(8)$$

lead, up to the second order in g, to the following expressions of the partition functions

$$Z_{2D} \approx Z_x^\omega Z_y^\omega \left[1 - \frac{g^2}{2\hbar^2}\int\limits_0^{\hbar\beta} d\tau_1 \int\limits_0^{\hbar\beta} d\tau_2 \left\langle x\left(\tau_1\right)y^2\left(\tau_1\right)x\left(\tau_2\right)y^2\left(\tau_2\right)\right\rangle_\omega\right], \quad (9)$$

$$Z_{3D} \approx Z_x^\omega Z_y^\omega Z_z^\omega \left[1 - \frac{g^2}{2\hbar^2} \int\limits_0^{\hbar\beta} d\tau_1 \int\limits_0^{\hbar\beta} d\tau_2 \left\langle x\left(\tau_1\right) y\left(\tau_1\right) z\left(\tau_1\right) x\left(\tau_2\right) y\left(\tau_2\right) z\left(\tau_2\right) \right\rangle_\omega \right],$$
(10)

where $Z_{x_\alpha}^\omega$ are the partition functions of the harmonic oscillator

$$Z_x^\omega = Z_y^\omega = Z_z^\omega = \frac{1}{2} \sinh^{-1} \frac{\hbar\beta\omega}{2}.$$
(11)

Expectation values in (9) and (10) can be performed by applying generalized Wick's rules

- each expectation values of a product of 2-paths defines the propagator

$$G_\omega\left(\tau_1, \tau_2\right) = \begin{cases} \left\langle x_i\left(\tau_1\right) x_j\left(\tau_2\right)\right\rangle_\omega \neq 0; & \text{for} \quad i = j. \\ \left\langle x_i\left(\tau_1\right) x_j\left(\tau_2\right)\right\rangle_\omega = 0; & \text{for} \quad i \neq j. \end{cases}$$
(12)

- each expectation values of a product of n-paths reads, taking into account the first rule (12)

$$\begin{aligned} \left\langle x_i\left(\tau_1\right) x_j\left(\tau_2\right) \cdots x_q\left(\tau_n\right)\right\rangle_\omega &= G_\omega\left(\tau_1, \tau_2\right) \left\langle x_k\left(\tau_3\right) x_l\left(\tau_4\right) \cdots x_q\left(\tau_n\right)\right\rangle_\omega \\ &\quad + G_\omega\left(\tau_1, \tau_3\right) \left\langle x_j\left(\tau_2\right) \cdots x_q\left(\tau_n\right)\right\rangle_\omega + \cdots \\ &\quad + G_\omega\left(\tau_1, \tau_n\right) \left\langle x_j\left(\tau_2\right) \cdots x_p\left(\tau_{n-1}\right)\right\rangle_\omega, \end{aligned}$$
(13)

By applying the Wick's rules and knowing that the free energy reads as $F = -k_B T \ln Z$, we derive the perturbation series at low temperatures $(T \to 0)$ for the ground-state energy of the potentials (2) and (3) up to the fourth-order by the connected Feynman diagrams

$$\begin{aligned} E_{2D} &= \hbar\,\omega - \lim_{T\to 0} k_B\, T \left[\frac{1}{72} \qquad\quad + \frac{1}{36} \quad + \frac{1}{324} \qquad\quad + \frac{5}{1296} \right. \\ &\quad \left. + \frac{1}{648} \quad + \frac{1}{1296} \qquad\qquad \right] + O\left(g^6\right), \end{aligned}$$
(14)

$$E_{3D} = \frac{3}{2}\,\hbar\,\omega - \lim_{T\to 0} k_B T \left[\frac{1}{72} \quad + \frac{1}{1728} \quad + \frac{1}{5184} \quad \right] + O\left(g^6\right).$$
(15)

For evaluating the connected Feynman diagrams, the following Feynman rules are introduced [2,14]:

- for propagator

$$\rightarrow \quad G_\omega\left(\tau_a, \tau_b\right) = \frac{\hbar}{2\omega} \exp\left[-\omega\left|\tau_a - \tau_b\right|\right]. \qquad (16)$$

- for vertices

$$\rightarrow \quad \frac{-6\ i\ g}{\hbar} \int_0^{\hbar\beta} d\tau_a. \qquad (17)$$

Applying the Feynman Rules to (4) and (5), we obtain the analytical expressions for the ground-state energy [10,12]

$$E_{2D}^{(0)} = \hbar\omega + \frac{5}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{223}{864}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \qquad (18)$$

$$E_{3D}^{(0)} = \frac{3}{2}\hbar\omega + \frac{1}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{7}{576}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \qquad (19)$$

3 Bender-Wu perturbation theory : Recursion relations

We derive here recursion relations of the perturbation coefficients of the ground-state energy for the potentials (2) and (3). These recursion relations are obtained from the corresponding Schrödinger equations

$$-\frac{\hbar^2}{2M}\nabla_{2D}^2\psi\left(x,y\right) + \left[\frac{M\omega^2}{2}\left(x^2 + y^2\right) + igxy^2\right]\psi\left(x,y\right) = E_{2D}\psi\left(x,y\right). \qquad (20)$$

$$-\frac{\hbar^2}{2M}\nabla_{3D}^2\psi\left(x,y,z\right) + \left[\frac{M\omega^2}{2}\left(x^2 + y^2 + z^2\right) + igxyz\right]\psi\left(x,y,z\right) = E_{3D}\psi\left(x,y,z\right). \qquad (21)$$

In the perturbation theory, the ground-state wave functions are expanded in the form [3-5,12]

$$\psi\left(x,y\right) = \mathcal{N}_0 \exp\left[-\frac{M\omega}{2\hbar}\left(x^2 + y^2\right) + \phi\left(x,y\right)\right]. \qquad (22)$$

$$\psi\left(x, y, z\right) = \mathcal{N}_0 \exp\left[-\frac{M\omega}{2\hbar}\left(x^2 + y^2 + z^2\right) + \phi\left(x, y, z\right)\right]. \qquad (23)$$

The wave functions $\phi\left(x_\alpha\right)$, where $\alpha = 1, 2, 3$, will be expanded in powers of the coupling constant g

$$\phi\left(x_\alpha\right) = \sum_{k=1}^{\infty} g^k \phi_k\left(x_\alpha\right). \qquad (24)$$

The ground-state energy corresponding to (2) and (3) can be expanded in powers of the coupling constant g

$$E_{2D} = \hbar\omega + \sum_{k=1}^{\infty} g^k \epsilon_k^{(2D)}. \qquad (25)$$

$$E_{3D} = \frac{3}{2}\hbar\omega + \sum_{k=1}^{\infty} g^k \epsilon_k^{(3D)}. \qquad (26)$$

Inserting (24), (25) and (26) into (22) and (23), we obtain a differential equation taking into account natural units ($\hbar = M = \omega = 1$)

$$
\begin{aligned}
\epsilon_k^{(2D)} \;=\; & -\frac{1}{2}\nabla_{2D}^2 \phi_k\left(x_\alpha\right) + x\,\partial_x \phi_k\left(x_\alpha\right) + y\,\partial_y \phi_k\left(x_\alpha\right) + igxy^2\delta_{k,1} \\
& -\frac{1}{2}\sum_{l=1}^{k-1}\left[\partial_x \phi_{k-l}\left(x_\alpha\right)\partial_x \phi_l\left(x_\alpha\right) + \partial_y \phi_{k-l}\left(x_\alpha\right)\partial_y \phi_l\left(x_\alpha\right)\right]. \quad (27)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_k^{(3D)} \;=\; & -\frac{1}{2}\nabla_{3D}^2 \phi_k\left(x_\alpha\right) + x\,\partial_x \phi_k\left(x_\alpha\right) + y\,\partial_y \phi_k\left(x_\alpha\right) + z\,\partial_z \phi_k\left(x_\alpha\right) \\
& +igxyz\delta_{k,1} - \frac{1}{2}\sum_{l=1}^{k-1}\left[\partial_x \phi_{k-l}\left(x_\alpha\right)\partial_x \phi_l\left(x_\alpha\right) + \partial_y \phi_{k-l}\left(x_\alpha\right)\partial_y \phi_l\left(x_\alpha\right)\right. \\
& \left.+\partial_z \phi_{k-l}\left(x_\alpha\right)\partial_z \phi_l\left(x_\alpha\right)\right], \qquad (28)
\end{aligned}
$$

where we have used the abbreviation $\partial_{x_\alpha} \equiv \dfrac{\partial}{\partial x_\alpha}$.

The $\phi_k(x_\alpha)$ are expanded in powers of the coordinates as [3-5,12]

$$\phi_k(x,y) = \sum_{j=0}^{k} \sum_{m=0}^{k} a_{j,m}^{(k)} x^j y^{2m}. \qquad (29)$$

$$\phi_k(x,y,z) = \sum_{j=0}^{k} \sum_{m=0}^{k} \sum_{n=0}^{k} a_{j,m,n}^{(k)} x^j y^m z^n. \qquad (30)$$

where $a_{j,m}^{(k)}$ and $a_{j,m,n}^{(k)}$ are non-symmetrical coefficients and can be real *and/or* imaginary.

By inserting (29) and (30) into (27) and (28), we obtain in second order the following coefficients

- 2-D

For $k = 1$,

$$a_{01}^{(1)} = 0, \quad a_{10}^{(1)} = a_{11}^{(1)} = -\frac{i}{3}. \qquad (31)$$

For $k = 2$,

$$\begin{aligned} a_{10}^{(2)} &= a_{11}^{(2)} = a_{12}^{(2)} = a_{22}^{(2)} = 0, \ a_{01}^{(2)} = -\frac{1}{8} \\ a_{20}^{(2)} &= -\frac{1}{36}, \ a_{21}^{(2)} = -\frac{1}{18}, \ a_{02}^{(2)} = -\frac{1}{72}. \end{aligned} \qquad (32)$$

- 3-D

For $k = 1$,

$$\begin{aligned} a_{100}^{(1)} &= a_{010}^{(1)} = a_{001}^{(1)} = a_{110}^{(1)} = a_{101}^{(1)} = a_{011}^{(1)} = 0, \\ a_{111}^{(1)} &= -\frac{i}{3}. \end{aligned} \qquad (33)$$

For $k = 2$,

$$a_{200}^{(2)} = a_{020}^{(2)} = a_{002}^{(2)} = a_{220}^{(2)} = a_{202}^{(2)} = a_{022}^{(2)} = -\frac{1}{72}, \text{ all others } a_{j,m,n}^{(2)} = 0. \qquad (34)$$

For $k \geq 3$, we find the recursion relations and the energy correction coefficients

8

- 2-D

$$a_{j,m}^{(k)} = \frac{1}{2(j+m)} \left[a_{j-1,m-1}^{(k-1)} - 2 \sum_{k'=1}^{k} \left(a_{2,0}^{(k')} + a_{0,2}^{(k')} \right) a_{j,m}^{(k-k')} \right.$$
$$\left. + (j+1)(j+2) a_{j+2,m}^{(k)} + (m+1)(m+2) a_{j,m+2}^{(k)} \right], (35)$$

$$\epsilon_0^{(2D)} = \frac{1}{2} (-1)^{k+1} \left(a_{2,0}^{(2k)} + a_{0,2}^{(2k)} \right). \tag{36}$$

- 3-D

$$a_{j,m,n}^{(k)} = \frac{1}{2(j+m+n)} \left[a_{j-1,m-1,n-1}^{(k-1)} - 2 \sum_{k'=1}^{k} \left(a_{2,0,0}^{(k')} + a_{0,2,0}^{(k')} + a_{0,0,2}^{(k')} \right) \right.$$
$$a_{j,m,n}^{(k-k')} + (j+1)(j+2) a_{j+2,m,n}^{(k)} + (m+1)(m+2) a_{j,m+2,n}^{(k)}$$
$$\left. + (n+1)(n+2) a_{j,m,n+2}^{(k)} \right] \tag{37}$$

$$\epsilon_0^{(3D)} = \frac{1}{2} (-1)^{k+1} \left(a_{2,0,0}^{(2k)} + a_{0,2,0}^{(2k)} + a_{0,0,2}^{(2k)} \right). \tag{38}$$

Table 1 and Table 2 show the coefficients up to the 10th order [10,12]; one sees that the results obtained by the Bender-Wu method are in agreement with those obtained by the connected fourth order Feynman diagrams (see (18) and (19)).

Table 1
Weak-coupling coefficients for the 2-dimensional potential up to the 10th order, k represents the order of expansion and ϵ_k coefficients of the energy corrections.

k	1	2	3	4	5
ϵ_k	0	$\frac{5}{24}$	0	$-\frac{223}{864}$	0
k	6	7	8	9	10
ϵ_k	$\frac{116407}{155520}$	0	$-\frac{346266143}{111974400}$	0	$\frac{2360833242959}{141087744000}$

Table 2

Weak-coupling coefficients for the 3-dimensional potential up to the 10th order, k represents the order of expansion and ϵ_k coefficients of the energy corrections.

k	1	2	3	4	5
ϵ_k	0	$\frac{1}{24}$	0	$-\frac{5}{576}$	0
k	6	7	8	9	10
ϵ_k	$\frac{5069}{622080}$	0	$-\frac{2441189}{289598400}$	0	$\frac{8034211571}{752467968000}$

4 Resummation procedure : Strong-coupling limit

In this section, we are interested to resume the perturbation series (18) and (19) to the strong-coupling limit. To this end, substituting the coordinate x_α, where $\alpha = 1, 2, 3$, by

$$x_\alpha \longrightarrow g^{-1/5} x_\alpha, \tag{39}$$

the Schrödinger equations (20) and (21) become

$$-\frac{\hbar^2}{2M}\nabla^2_{2D}\psi\left(x,y\right)+\left[\frac{M\,\omega^2}{2}g^{-4/5}\left(x^2+y^2\right)+ixy^2\right]\psi\left(x,y\right)=g^{-4/5}E_{2D}\psi\left(x,y\right). \tag{40}$$

$$-\frac{\hbar^2}{2M}\nabla^2_{3D}\psi\left(x,y,z\right)+\left[\frac{M\,\omega^2}{2}g^{-4/5}\left(x^2+y^2+z^2\right)+ixyz\right]\psi\left(x,y,z\right)$$
$$= g^{-4/5}E_{3D}\psi\left(x,y,z\right). \tag{41}$$

The wave functions and the energy will be expanded in powers of the coupling constant g and yield [13,15]

$$\psi\left(x_\alpha\right)=\psi_0\left(x_\alpha\right)+g^{-4/5}\psi_1\left(x_\alpha\right)+g^{-8/5}\psi_2\left(x_\alpha\right)+\cdots. \tag{42}$$

$$E_D=g^{2/5}\left(\epsilon_0+g^{-4/5}\epsilon_1+g^{-8/5}\epsilon_2+\cdots\right). \tag{43}$$

The strong-coupling coefficients of the ground-state for the potentials (2) and (3) can be obtained by resuming the weak-coupling series [10,12] obtained in the last two sections. Up to the fourth order, the weak-coupling series read

$$E_{2D}^{(0)} = \hbar\omega + \frac{5}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{223}{864}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \tag{44}$$

$$E_{3D}^{(0)} = \frac{3}{2}\hbar\omega + \frac{1}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{7}{576}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \tag{45}$$

The power behavior of $E_D^{(0)}$ taken in (43) is independent of the order N and solely the coefficients depend on N

$$E_D^{(N)} = g^{2/5}\left(\epsilon_0^{(N)} + g^{-4/5}\epsilon_1^{(N)} + g^{-8/5}\epsilon_2^{(N)} + \cdots\right). \tag{46}$$

In order to perform a resummation, an artificial parameter trick is introduced often called Kleinert's square-root [2,13-15]

$$\omega \to \Omega\sqrt{1 + g^2 r}, \tag{47}$$

with

$$r = \frac{\omega^2 - \Omega^2}{g^2\Omega^2}. \tag{48}$$

Therefore, inserting (47) into (44) and (45) and re-expanding in coupling constant g^2 to the first order taking into account (48), we obtain

$$E_{2D}^{(1)}\left(g, \omega, \Omega\right) = \frac{\hbar\Omega}{2} + \frac{\hbar\omega^2}{2\Omega} + g^2\frac{5\hbar^2}{24\Omega^4} + O\left(g^4\right). \tag{49}$$

$$E_{3D}^{(1)}\left(g, \omega, \Omega\right) = \frac{3\hbar\Omega}{2} + \frac{3\hbar\omega^2}{4\Omega} + g^2\frac{\hbar^2}{24\Omega^4} + O\left(g^4\right). \tag{50}$$

Performing the first derivative with respect to the variational parameter Ω and considering its strong-coupling behavior [13,15]

$$\Omega = g^{2/5}\left(\Omega_0 + g^{-4/5}\Omega_1 + g^{-8/5}\Omega_2 + \cdots\right) \tag{51}$$

we get

- 2-D

$$\Omega_0 = \sqrt[5]{\frac{5\hbar}{3}} \quad ; \quad \Omega_1 = \frac{\omega^2}{5} \sqrt[5]{\frac{3}{5\hbar}} \quad ; \quad \Omega_2 = \frac{\omega^4}{25} \sqrt[5]{\frac{27}{125\hbar^3}}. \qquad (52)$$

- 3-D

$$\Omega_0 = \sqrt[5]{\frac{2\hbar}{9}} \quad ; \quad \Omega_1 = \frac{\omega^2}{5} \sqrt[5]{\frac{9}{2\hbar}} \quad ; \quad \Omega_2 = \frac{\omega^4}{25} \sqrt[5]{\frac{3}{8\hbar^3}}. \qquad (53)$$

Inserting these results into (49) and (50) yield the strong-coupling coefficients of the ground-state energy $\epsilon_k^{(N)}$

- 2-D

$$\epsilon_0^{(1)} = \frac{5}{8} \sqrt[5]{\frac{5\hbar^6}{3}} \approx 0.69222\hbar^{6/5}; \quad \epsilon_1^{(1)} = \frac{\omega^2}{2} \sqrt[5]{\frac{3\hbar^4}{5}}; \quad \epsilon_2^{(1)} = \frac{\omega^4}{20} \sqrt[5]{\frac{3\hbar^2}{5}}. \qquad (54)$$

- 3-D

$$\epsilon_0^{(1)} = \frac{5}{8} \sqrt[5]{\frac{27\hbar^6}{16}} \approx 0.69395\hbar^{6/5}; \quad \epsilon_1^{(1)} = \frac{3\omega^2}{4} \sqrt[5]{\frac{9\hbar^4}{2}}; \quad \epsilon_2^{(1)} = -\frac{9\omega^4}{40} \sqrt[5]{\frac{3\hbar^2}{8}}. \qquad (55)$$

However, it turns out that the numerical values of the leading strong-coupling coefficients obtained in (54) and (55) are lower than those calculated in weak-coupling limit (see (18) and (19)). Thus, the convergence of the VPT-results is less satisfactory. Following [13], we must rely upon that the farfetched values must be higher than those calculated in weak-coupling limit. It is what we are going to show in section 5.

5   D-dimensional complex effective potential

For any given potential, the corresponding effective potential is obtained by performing a Legendre transformation of the ground-state energy in the special case of the external current being constant [2,15]; it will be

expanded in powers of $\hbar$ rather than the coupling constant g and depends on the new parameter X_α, where $\alpha = 1, 2, 3$, namely the path average,

$$
\begin{aligned}
V_{\text{eff}}(X_\alpha) & \equiv \sum_{l=0}^{\infty} \hbar^l V^{(l)}(X_\alpha) \\
& = V(X_\alpha) + \frac{\hbar}{2} \sum_\alpha \operatorname{tr} \ln G_{X_\alpha}^{-1} + V_D^{(\text{int})}(X_\alpha).
\end{aligned}
\tag{56}
$$

where the superscript l indicates the number of loops involved in Feynman diagrams, the trace-logarithm functions are given by the ground-state energy of harmonic oscillators, they are connected to the partial frequencies by

$$
\begin{aligned}
\frac{\hbar}{2} \operatorname{tr} \ln G_{X_\alpha}^{-1} & \equiv \frac{\hbar \widetilde{\omega}_{X_\alpha}}{2} \\
& = \frac{\hbar}{2} \sqrt{\partial_{X_\alpha}^2 V_D(X_\alpha)}.
\end{aligned}
\tag{57}
$$

The frequency of the propagator is given as the sum of all partial frequencies

$$
\widetilde{\omega}_D = \sqrt{\sum_\alpha \widetilde{\omega}_{X_\alpha}^2}.
\tag{58}
$$

$V^{(\text{int})}(X_\alpha)$ is called interaction potential [2,13-15] and contains all one-particle irreducible vacuum diagrams.

5.1   Weak-coupling limit

The aim of this section is to deduce weak-coupling ground-state energy for the potentials (2) and (3) by using the effective potential [10,12]. The computation will be performed until the third-loop order at low temperatures, i.e. $(T \rightarrow 0)$.

The frequency of the propagator of the potentials (2) and (3) are now given, using (57) and (58), by

$$
\widetilde{\omega}_{2D} = \sqrt{2\omega^2 + 2igX}.
\tag{59}
$$

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$$\widetilde{\omega}_{3D} = \sqrt{3}\,\omega. \tag{60}$$

The corresponding interaction potentials are read

$$V_{2D}^{(\text{int})}(X,Y) = -\lim_{T\to 0} k_B T \left[\frac{1}{36} \quad + \frac{5}{1296} \quad + \frac{1}{648} \quad \right] + O\left(\hbar^4\right). \tag{61}$$

$$V_{3D}^{(\text{int})}(X,Y,Z) = -\lim_{T\to 0} k_B T \left[\frac{1}{72} \quad + \frac{1}{1728} \quad + \frac{1}{5184} \quad \right] + O\left(\hbar^4\right). \tag{62}$$

where the previous diagrams are deduced from new-Feynman laws

- for propagator

$$\rightarrow \quad G_\omega\left(\tau_a, \tau_b\right). \tag{63}$$

- for vertices

$$\rightarrow \quad \frac{-D}{\hbar} \sum_{\alpha=i}^{D} \sum_{\alpha=j}^{D} \sum_{\alpha=k}^{D} \partial_i \partial_j \partial_k \, V\left(X_\alpha\right). \tag{64}$$

Substituting (59)-(62) into (56) and taking into account (63) and (64), we obtain the expressions of effective potential at low temperatures

$$\lim_{T\to 0} V_{\text{eff}}^{(2D)}(X,Y) = \frac{\omega^2}{2}\left(X^2+Y^2\right) + igXY^2 + \frac{\hbar}{2}\omega + \frac{\hbar}{2}\sqrt{\omega^2 + 2igX} +$$

$$\frac{\hbar^2 g^2}{3\left(2\omega^2 + 2igX\right)^{3/2}} - \left[\frac{2}{3}\frac{1}{648} + \frac{22}{27}\frac{5}{1296}\right]\frac{324\sqrt{2}\hbar^3 g^4}{\left(2\omega^2 + 2igX\right)^{9/2}} + O\left(\hbar^4\right). \tag{65}$$

$$\lim_{T\to 0} V_{\text{eff}}^{(3D)}(X,Y,Z) = \frac{\omega^2}{2}\left(X^2+Y^2+Z^2\right) + igXYZ + \frac{3\hbar\omega}{2} + \frac{\hbar^2 g^2}{24\omega^4}$$

$$- \left[\frac{2}{3}\frac{1}{5184} + \frac{22}{27}\frac{5}{1728}\right]\frac{81\hbar^3 g^4}{4\omega^9} + O\left(\hbar^4\right). \tag{66}$$

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The path average X_α in the case of complex potentials can be expanded in the form [13,15]

$$X_\alpha = i \ \left(X_{\alpha 0} + \hbar X_{\alpha 1} + \hbar^2 X_{\alpha 2} + \cdots \right). \tag{67}$$

Inserting this identity into (65) and (66), and performing the first derivative with respect to X_α and setting the resulting expressions to zero, we get

- 2-D

$$X_{10} \ \equiv \ X_0 = 0; \quad X_{11} \equiv X_1 = -\frac{g}{2\omega^3}; \quad X_{12} \equiv X_2 = \frac{5g^3}{12\omega^3} \tag{68.a}$$

$$X_{20} \ \equiv \ Y_0 = 0; \quad X_{21} \equiv Y_1 = 0; \quad\quad X_{22} \equiv Y_2 = 0. \tag{68.b}$$

- 3-D

$$X_{\alpha k} \equiv 0. \tag{69}$$

for $\alpha = 1, 2, 3$, and $k = 0, 1, 2$.

Re-inserting (68) and (69), respectively, into (65) and (66), and expanding until the third–order in $\hbar$ yields the ground-state energy at low temperatures

$$E_{2D}^{(0)} = \hbar\omega + \frac{5}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{223}{864}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \tag{70}$$

$$E_{3D}^{(0)} = \frac{3}{2}\hbar\omega + \frac{1}{24}\frac{g^2\hbar^2}{M^3\omega^4} - \frac{7}{576}\frac{g^4\hbar^3}{M^6\omega^9} + O\left(g^6\right). \tag{71}$$

The results are in agreement with those obtained in sections 2 and 3.

5.2   Strong-coupling limit

Let us now resume the effective potential. Since the effective potential is expanded in powers of $\hbar$, Kleinert's square-root will be converted accordingly to [13-15]

$$\omega \to \Omega\sqrt{1 + \hbar\, r}. \tag{72}$$

with

$$r = \frac{\omega^2 - \Omega^2}{\hbar\,\Omega^2}. \tag{73}$$

5.2.1 Two-dimensional case

First-order Substituting (72) and (73) into (65) at first order in $\hbar$, we obtain

$$V_{\text{eff},1}^{(2D)}(X,Y) = \frac{\omega^2}{2}\left(X^2 + Y^2\right) + igXY^2 + \frac{\hbar\Omega}{2} + \frac{\hbar}{2}\sqrt{\Omega^2 + 2igX}. \quad (74)$$

We now optimize in Ω and X_α; the resulting equations allow us to determine the strong-coupling behavior of X_α given by [13,15]

$$X_\alpha = -ig^{-1/5}\left(X_{\alpha 0} + X_{\alpha 1}g^{-4/5} + X_{\alpha 2}g^{-8/5} + \cdots\right). \quad (75)$$

where the corresponding coefficients read as

$$\begin{aligned} X_0 &= 0.417288\ \hbar^{6/5}; \ X_1 = -0.245404\ \omega^2; \ X_2 = 0.063624\ \omega^2\hbar^{-6/5} && (76.\text{a})\\ Y_k &= \sqrt{2}\ X_k; \qquad k = 0, 1, 2. && (76.\text{b}) \end{aligned}$$

Re-inserting the results (76) into (75), and again in (74), we obtain the strong-coupling behavior of the ground-state energy

$$\epsilon_0^{(1)} \approx 1.1263168\ \hbar^{6/5}. \quad (77)$$

Second order In second order, the effective potential becomes

$$\begin{aligned} V_{\text{eff},2}^{(2D)}(X,Y) &= \frac{\omega^2}{2}\left(X^2 + Y^2\right) + igXY^2 + \frac{\hbar}{4}\left[\frac{\omega^2}{\Omega^2} + \Omega + \frac{\omega^2 + \Omega^2 + 4igX}{\sqrt{\Omega^2 + 2igX}}\right]\\ &\quad + \frac{g^2\hbar^2}{12\left(\Omega^2 + igX\right)^2}. \end{aligned} \quad (78)$$

Following the same steps as was done in the first order, the strong-coupling behavior of the ground-state energy in second order is

$$\epsilon_0^{(2)} \approx 1.13595605\ \hbar^{6/5}. \quad (79)$$

5.2.2 Three-dimensional case

First order Substituting (72) and (73) into (66) at first order in $\hbar$, we obtain the effective potential

$$V_{\text{eff},1}^{(3D)}(x) = \frac{\omega^2}{2}\left(X^2 + Y^2 + Z^2\right) + igXYZ + \frac{3\hbar\Omega}{2}. \tag{80}$$

Let us now optimize in Ω and X_α, with $\alpha = 1, 2, 3$. We can note that, from (80), the first derivative with respect to Ω is constant $\left(= \dfrac{3\hbar}{2}\right)$ and different than zero. In order to avoid this ambiguity, we need to substitute Ω in (80) by a mathematical trick according to

$$\Omega \to \sqrt{\Omega^2 + 2\, ig\lambda X}, \tag{81}$$

where $\lambda \ll 1$ and by performing λ-expansion around zero $(\lambda \to 0)$, we obtain

$$X_0 = \frac{1}{\lambda}\sqrt[5]{\frac{\hbar^2}{648}}; \qquad X_1 = \frac{\omega^2}{5\lambda}; \qquad X_2 = \frac{\omega^2\left(20 + \sqrt[5]{648}\,\omega^2\right)}{100\lambda}\hbar^{-6/5} \tag{82.a}$$

$$Y_0 = 0; \qquad\qquad Y_1 = \omega^2; \qquad Y_2 = \hbar^{-6/5}. \tag{82.b}$$

$$Z_0 = -\frac{1}{\lambda}\sqrt[5]{\frac{\hbar^2}{648}}; \quad Z_1 = -\frac{\omega^2}{5\lambda}; \qquad Z_2 = -\frac{3\omega^4}{50\lambda}\sqrt[5]{\frac{81}{5\hbar^6}}. \tag{82.c}$$

Re-inserting (81) and (82) into (80), taking into account (75), we obtain the strong-coupling behavior of the ground-state energy at first-order

$$\epsilon_0^{(1)} = \frac{3^{3/5}}{2^{3/10}}\,\hbar^{6/5} \approx 1.5702317\,\hbar^{6/5}. \tag{83}$$

Second order Substituting (72) and (73) into (66), and expanding the result until the second–order in $\hbar$ yields the effective potential

$$V_{\text{eff},2}^{(3D)}(X, Y, Z) = \frac{\omega^2}{2}\left(X^2 + Y^2 + Z^2\right) + igXYZ + \frac{3\left(\omega^2 + \Omega^2\right)\hbar}{4\Omega} + \frac{\hbar^2}{24\Omega^4}. \tag{84}$$

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Again, the strong-coupling behavior of the ground-state energy in second order is obtainable following the same steps as was done in the first order, we obtain

$$\epsilon_0^{(2)} = \frac{3^{3/5}}{2^{3/10}} \frac{\sqrt{1 + \sqrt{230}}}{4} \, \hbar^{6/5} \approx 1.5783441 \, \hbar^{6/5}. \tag{85}$$

In both cases, one can sees that the new values of the leading strong-coupling coefficients are in good agreement with those expected with a small deviation from the first order, i.e. $\dfrac{\left| \epsilon_0^{(2)} - \epsilon_0^{(1)} \right|}{\epsilon_0^{(1)}}$, in order of 0.8% and 0.5%, respectively.

6 Conclusion

In sections 2 and 3, we have derived the weak-coupling coefficients for the ground-state energy for the potentials (2) and (3) using the Feynman diagrammatical expansion and Bender-Wu recursion relations. Both results are in agreement with those obtained by the old-fashion perturbation theory, namely the Rayleigh-Schrödinger method. We have proceeded in section 4 to resume the weak-coupling series (70) and (71) to the strong-coupling limit by applying Kleinert's square-root trick. However, the leading strong-coupling coefficients for the ground-state energy lie below those calculated in weak-coupling limit, thus, the rate of convergence is less satisfactory. We introduce in section 5 the effective potential which, once combined with the VPT, allows to determine the same weak-coupling series obtained in sections 2 and 3. In order to recover the rate of convergence in the strong-coupling limit, we proceed to resume the effective potential. It turns out that VPT-results for leading strong-coupling coefficients obtained in second order (79) and (85) approaches those expected with a very small deviation compared to the first order (77) and (83), thanks to the introduction of the path average variational parameter.

Following [16], it is interesting to apply the covariant effective potential for particle moving in one-dimensional complex cubic potential for solving the corresponding Schrödinger equation with position-dependant mass.

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